of coefficients for $y^+ > 30$. However such segmental representations produce discontinuities in du/dy at the intersections which in turn result in very awkward expressions for heat and component transfer.

Many functions with continuous derivatives have been proposed to represent the velocity distribution from the wall to the centerline. Most of these suffer from inaccuracy or complexity.

The general expression proposed by Churchill and Usagi (1972) for correlation of rate data can be used to construct an empirical equation which interpolates continuously between Equations (1) and (2). The trial expression is

$$\frac{y^+}{u^+} = \left[1 + \left(\frac{y^+}{A + B \ln y^+}\right)^n\right]^{1/n} \quad (3)$$

Representative experimental data from the work of Abbrecht and Churchill (1960) are plotted in the suggested form in Figure 1 with $A = 5.5$ and $B = 2.5$. A value of $n = 2$ is seen to provide a reasonable overall fit yielding a correlation which can be rearranged as

$$\frac{1}{u^{+2}} = \frac{1}{y^{+2}} + \frac{0.16}{\ln^2(9y^+)} \quad (4)$$

The coordinates of Figure 1 exaggerate the scatter of the data and the deviations from Equation (4). The more conventional coordinates of Figure 2 confirm that a very good representation is provided by Equation (4).

Equation (4) is in itself somewhat unwieldy in form but: (1) it is continuous and has continuous derivatives for $y^+ > 0.11$; (2) it approaches the two well-known limiting solutions asymptotically; (3) it does not require integration or the evaluation of complex functions and (4) it is convenient for slide rule or machine calculations. The singularity at $y^+ \cong 0.11$ is not of practical concern in calculations since Equation (1) can be used rather than Equation (4) for $y^+ < 5$.

NOTATION

A = empirical coefficient [Equation (2)], dimensionless
B = empirical coefficient [Equation (2)], dimensionless
n = empirical exponent [Equation (3)], dimensionless
u = velocity, m/s
u^+ = $u/\sqrt{\tau_w/\rho}$, dimensionless
y = distance from wall, m
y^+ = $y\sqrt{\tau_w/\rho}/\nu$
ν = kinematic viscosity, m^2/s
ρ = density, kg/m^3
τ_w = shear stress on wall, N/m^2

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Theoretical Non-Newtonian Pipe-Flow Heat Transfer

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Numerous investigations of turbulent non-Newtonian pipe-flow heat transfer have resulted in correlations of heat transfer to purely viscous fluids based on empirically fitting turbulent heating data or on the analogy between energy and momentum transport. Correlations of momentum transport are then used in the latter case to provide predictions of heat transfer. The present analysis is based on integration of the momentum and energy equations across the pipe diameter to yield a general solution to the enthalpy profiles and heat transfer for a power-law fluid under constant wall flux conditions. The velocity profiles are calculated using the author's formulation for turbulent momentum exchange. The objective of the study is to illustrate the effectiveness of a theoretical model in predicting non-Newtonian heat transfer in a way analogous with the theoretical treatment of Newtonian fluid flow.

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Metzner and Friend (1959) provided the first semitheoretical analysis of the problem based on the work of Dodge and Metzner and their previous work on turbulent pipe flow of Newtonian fluids (Friend and Metzner, 1958). The approach, based on analogy between heat and momentum exchange, permitted extension of the treatment of non-Newtonian fluids beyond the limited ranges dealt with previously on a purely empirical basis.

Clapp investigated the problem experimentally and developed semi-empirical temperature profiles as a function of pipe radius. In doing so, he formulated the local shear stress as the sum of the power law non-Newtonian shear and the purely turbulent shear based on Prandtl's momentum transfer theory. This is in analogy with Prandtl's approach for a Newtonian fluid. Wells (1968) extended the analogy between energy and momentum transport to include drag reducing fluids using a correlation of the viscous sublayer thickness and friction factor for dilute solutions of polymers (Meyer, 1966). Wells makes it clear that any treatment of the energy transfer of general non-

Newtonian systems must include a momentum transport theory which allows for departure from the Newtonian viscous sublayer and frictional characteristics.

Dodge and Metzner (1959) performed the first theoretical analysis for turbulent velocity profiles and drag characteristics of non-Newtonian fluid flow. Subsequent papers (Elata et al., 1966; Ernst, 1966, 1967; Granville, 1968; Meyer, 1966; Van Driest, 1970; Walters and Wells, 1971; Wells, 1965, 1970) have provided experimental data and empirical formulations of drag reduction in dilute elastoviscous polymer solutions. It has become clear from these references that conflicting interpretations of drag reduction have developed based on whether the mixing length parameters remains constant at 0.4 or is reduced by elastic effects within the fluid.

The author (Hecht, 1972) has analyzed the drag reduction in non-Newtonian power law fluids based on reduction of the mixing length parameter, an approach which has met with success in predicting anomalous velocity profile and skin friction measurements where the reduction in mixing length parameter was reported (Wells, 1965). Unfortunately, experimentally determined values of the mixing length parameter are not available in general, and to the author's knowledge, are completely absent in the non-Newtonian experimental heat transfer literature.

ANALYSIS

The ratio of local heat transfer in a pipe to the value at the wall may be written for constant wall heat flux as

$$q/q_w = 2/V(r/R) \int_0^{r/R} u\ (r/R)\ d\ (r/R) \qquad (1)$$

where V is the mean linear velocity. Definition of the variables on the right-hand side of (1) require further investigation of the velocity profiles. Note that if the velocity across the pipe is assumed uniform and equal to V, Equation (1) results in a linear variation of q/q_w, an assumption made by Metzner and Friend.

The turbulent shear stress is written in the conventional form

$$\tau = (\mu_e + \rho\epsilon)\ du/dy \qquad (2)$$

where ϵ is the eddy viscosity and μ_e is an effective non-Newtonian laminar viscosity. The heat transfer is written

$$q = \rho\ (\alpha_e + \epsilon_\alpha)\ dh/dy \qquad (3)$$

where h is the fluid enthalpy, ϵ_α is the eddy diffusivity, and α_e is an effective non-Newtonian laminar diffusivity.

By setting the ratio ϵ/ϵ_α equal to the turbulent Prandtl number and employing a linear variation of shear stress across the pipe radius, the enthalpy gradient may be written

$$dh/dy = N_{Pr}\, q\ (du/dy) / [\mu_e du/dy (1 - N_{Pr}/N_{Prt}) + \tau_w\ (N_{Pr}/N_{Prt}) (1 - y/R)] \qquad (4)$$

where the non-Newtonian laminar Prandtl number and viscosity are defined by

$$N_{Pr} = \eta^{1/N}\, C_p/(k\, \tau_w{}^{(1-N)/N}); \quad \mu_e = \eta^{1/N}/\tau_w{}^{(1-N)/N} \qquad (5)$$

The nondimensional parameters u_*, φ_*, y_*, and R_* as defined in the Nomenclature are now introduced into Equation (4) to yield

$$d\varphi_*/dy_* = N_{Pr}\ (q/q_w)\ du_*/dy_* / [N_{Pr}/N_{Prt} (1 - y_*/R_*) + (1 - N_{Pr}/N_{Prt})\ du_*/dy_*] \qquad (6)$$

Equation (1) is nondimensionalized to provide the ratio q/q_w to be used in Equation (6):

$$q/q_w = \left[V_* - 2 \int_0^{y_*/R_*} u_*\ (1 - y_*/R_*)\ d\ (y_*/R_*) \right] \Big/ [V_*\ (1 - y_*/R_*)] \qquad (7)$$

where $V_* = V\kappa/v_*$. κ is the mixing length parameter, taken to be equal to 0.4 for Newtonian fluids, and v_* is the friction velocity.

The turbulent shear stress is now written as the sum of the laminar non-Newtonian shear and the fully turbulent contribution:

$$\tau = \tau_w\ (1 - y/R) = \eta (du/dy)^N + \rho l^2\ (du/dy)^2 \qquad (8)$$

which in nondimensional form may be expressed as

$$(1 - y_*/R_*) = (du_*/dy_*)^N + l_*^2 (du_*/dy_*)^2 \qquad (9)$$

Equation (9) was solved numerically by the author (Hecht, 1972) to determine u_* as a function of y_* for specified values of the power law index. The mixing length in nondimensional form was written as

$$l_* = y_*\ [1 - \exp\ (-\ y_*/A_*)], \quad y_* \lesseqgtr (\lambda/\kappa)\ R_* \qquad (10a)$$

$$l_* = (\lambda/\kappa)\ R_*, \quad (\lambda/\kappa) R_* \lesseqgtr y_* \lesseqgtr R_* \qquad (10b)$$

where

$$\lambda = 0.09\ N^{3/4}/(3N + 1) \qquad (11)$$

and $A_* = 10.4$. The reader is referred to the author's paper for a detailed derivation of these equations.

Using the velocity distribution calculated from Equations (9) to (11) in Equation (7), and employing the resulting distribution of (q/q_w) in Equation (6) along with the now known distribution of (du_*/dy_*) yields an equation which may be integrated to determine the variation of φ_*. Defining a bulk value of φ_* such that

$$\Phi_* = (2/V_*) \int_0^1 u_*\, \varphi_*\ (1 - y_*/R_*)\ d\ (y_*/R_*) \qquad (12)$$

the Stanton number may be determined from

$$N_{St} = \kappa^2/(V_*\ \Phi_*) \qquad (13)$$

The method of determining non-Newtonian heat transfer as outlined above is fully analogous to the theoretical treatment of Newtonian heat transfer and reduces to the Newtonian case for $N = 1$.

Comparison of the results given by the present method and the Stanton number calculated from an equation developed by Metzner and Friend is given in Figures 1 and 2. Metzner and Friend's results are calculated from

$$N_{St} = (f/2)/[1.2 + 11.8\ (f/2)^{1/2}\ (N_{Pr} - 1)\ N_{Pr}{}^{-1/3}] \qquad (14)$$

The values of the mixing length parameter and the non-Newtonian power law index used in calculating the results of the present method are $\kappa = 0.4$, $N = 1.0$ (Figure 1), and $N = 0.5$ (Figure 2). The turbulent Prandtl number has been set equal to 1.0 for both figures.

Figure 3 presents a comparison of the theory with experimental data compiled by Deissler for both mass and heat transport in Newtonian fluids. The variation of the Stanton number with Prandtl (or Schmidt) number is shown at a Reynolds number of 50,000. This plot is included merely to illustrate the accuracy of the present

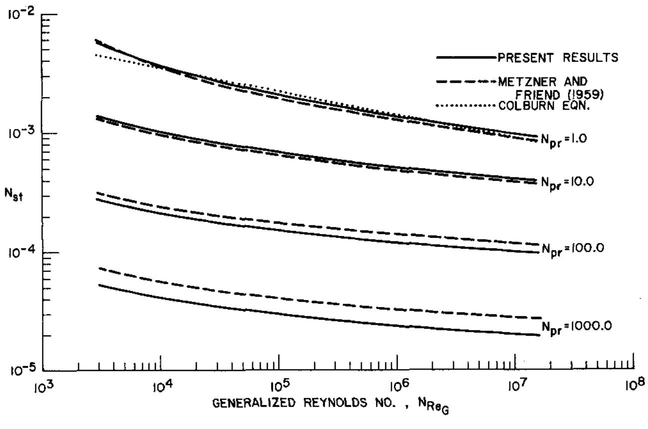

Fig. 1. Comparison of calculated Stanton number with the correlation of Metzner and Friend: $N = 1.0$, $N_{Prt} = 1.0$, $\kappa = 0.4$.

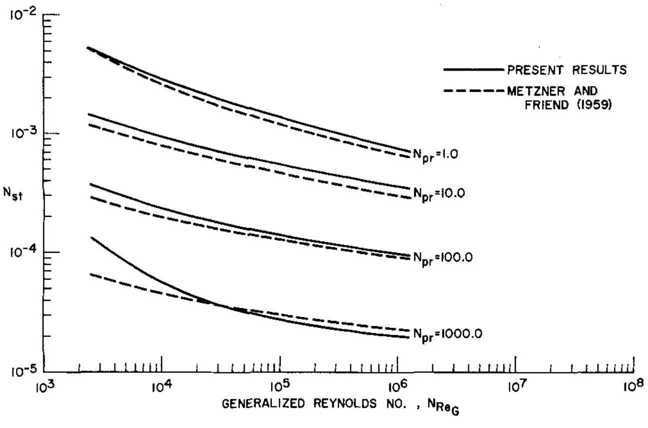

Fig. 2. Comparison of calculated Stanton number with the correlation of Metzner and Friend: $N = 0.5$, $N_{Prt} = 1.0$, $\kappa = 0.4$.

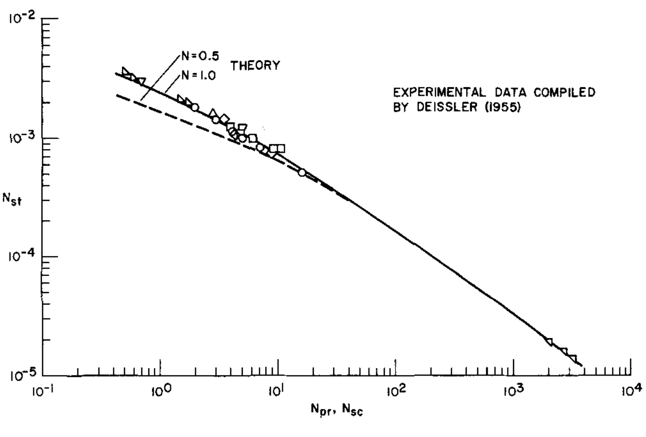

Fig. 3. Calculated Stanton number vs. Prandtl number compared to experimental data, $N_{ReG} = 5 \times 10^4$, $\kappa = 0.4$.

method for a Newtonian fluid, $N = 1.0$, $\kappa = 0.4$. The theoretical behavior of N_{St} with N_{Pr} for $N = 0.5$ is also shown in Figure 3.

NOTATION

A = damping factor (= 26 for Newtonian fluid)
A_* = $A\ \tau_w^{(2-N)/2N}\ \rho^{1/2}/\eta^{1/N}$
f = friction factor, $2\tau_w/\rho V^2$
h = enthalpy
k = fluid thermal conductivity
l = mixing length
N = power law index
N_{Pr} = laminar non-Newtonian Prandtl number,

$$\eta^{1/N}\ C_p/k\ \tau_w^{(1-N)/N}$$

N_{Prt} = turbulent Prandtl number
N_{Sc} = Schmidt number
N_{St} = Stanton number
N_{ReG} = generalized Reynolds number,

$$8[(2R)^N V^{2-N}\ \rho/\eta]/(6 + 2/N)^N$$

q = heat flux
r = radius
R = pipe radius
R_* = $R\ \kappa\ v_*^{(2-N)/N}\ \rho^{1/N}/\eta^{1/N}$
u = fluid velocity
u_* = $u\ \kappa/v_*$
V = mean velocity of flow
v_* = friction velocity, $(\tau_w/\rho)^{1/2}$
y = distance from tube wall
y_* = $y\ \kappa\ v_*^{(2-N)/N}\ \rho^{1/N}/\eta^{1/N}$
κ = mixing length parameter in wall region
λ = mixing length parameter in turbulent core
μ_e = non-Newtonian laminar viscosity, $\eta^{1/N}/\tau_w^{(1-N)/N}$
ρ = fluid density
τ = local shear stress
φ_* = enthalpy parameter, $(h - h_w)(\tau_w\rho)^{1/2}\ \kappa/q_w$
Φ_* = bulk enthalpy parameter

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